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Vibration of open circular cylindrical shells with intermediate ring supports

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Abstract

This paper is concerned with the free vibration of open circular cylindrical shells with intermediate ring supports. An analytical procedure for determining the free vibration frequencies of such shells is developed based on the Flügge thin shell theory. An open circular cylindrical shell is assumed to be simply supported along the two straight edges and the remaining two opposite curved edges may have any combinations of support conditions. The shell is divided into multiple segments along the locations of the intermediate ring supports. The state-space technique is employed to derive the exact solutions for each shell segment and the domain decomposition method is applied to enforce the geometric and natural boundary/interface conditions along the interfaces of the shell segments and the curved edges of the shell. Comparison studies are carried out to verify the correctness of the proposed method. Exact vibration frequencies are obtained for open circular cylindrical shells with multiple intermediate ring supports.

The influence of the number of intermediate ring supports, the locations of the ring supports, the boundary conditions and the variation of the included angle of the shells on the natural frequencies are examined. The exact vibration solutions can be used as important benchmark values for researchers to check their numerical methods and for engineers to design such shell structures.

Keywords: Open circular cylindrical shells; Intermediate ring supports; Flügge thin shell theory; Vibration; Analytical method; State-space technique; Exact solution

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1. Introduction

Shells and shell-type structures are important structural components in diverse engineering fields, i.e., civil, mechanical, aerospace, marine and offshore engineering. Open circular cylindrical shells are often applied as protective tank walls, thick cylindrical covers and roof structures of large-span and open-space buildings etc. It is very important for engineers to understand the vibration behaviour of such shell structures for more reliable and cost effective designs. However, the authors have found that rather little attention has been received on open cylindrical shells with intermediate supports in published literature as compared to those on closed cylindrical shells.

Vibration of open cylindrical shells has been studied by several researchers in the past few decades. An excellent collection of works on vibration of shells was published by Leissa (1973). In general, the finite element method was popular in solving the vibration problems of shells (Gontkevitch, 1962; Mustapha and Ali, 1987; Petyt, 1971; Sabir and Lock, 1972; Bardell et al., 1997). Boyd (1969) carried out a study on a simply supported open non-circular cylindrical shell based on Donnell shell theory. Suzuki and Leissa (1985, 1986) studied the free vibrations of circular and non-circular open cylindrical shells with circumferentially varying thickness. Srinivasan and Bobby (1976) developed a matrix method using Green's functions to analyse clamped cylindrical shell panels. Cheung et al. (1980) studied a singly curved shell panel and the spline finite strip method was employed in their studies. Lim et al. (1998) investigated the free vibration characteristics of thick and open shells based on a three-dimensional elasticity approach. Ohga et al. (1995) analysed the vibration of open cylindrical shells with a circumferential thickness taper by the matrix method. Selmane and Lakis (1997) explored the static and dynamic analysis of thin, elastic, anisotropic and non-uniform open cylindrical shells. Yu et al. (1995) used several analytical methods to examine the free vibration of open circular cylindrical shells with arbitrary combinations of simple boundary conditions. And the method of superposition introduced by Gorman (1982) was further developed and applied to open circular cylindrical shells in their studies (Yu et al., 1995). Three different shell theories were employed to study a cylindrical pipe and open shells by Price et al. (1998) and only simply supported boundary condition was considered in their investigation. Al-Jumaily and Ahmed (1985) used a simplified shell theory to investigate closed form solutions for the free vibrational characteristics of open profile circular cylindrical shells with different boundary conditions.

This paper aims to develop an analytical method based on the state-space technique for the vibration of open circular cylindrical shells with intermediate ring supports and to present exact vibration frequencies for such shells. The state-space technique associated with the Levy-type plate problems was extensively used by several researchers to obtain exact solutions for the buckling and vibration of rectangular plates (Khdeir, 1988; Reddy and Khdeir, 1989; Chen and Liu, 1990; Xiang and Liew, 1996; Liew et al., 1996). This technique has also been employed by the second author and his associates to study the buckling and vibration of plates with internal line supports and step-wise thickness variations (Xiang and Wei, 2002; Xiang and Wang, 2002). The authors have presented a preliminary study on the vibration of open cylindrical shells with intermediate ring supports (Zhang and Xiang, 2004). The present study employs the Flügge shell theory and an analytical method based on the state-space technique for circular cylindrical shells developed by the second author and his associates (Xiang et al., 2002) is further extended to study open cylindrical shells with intermediate ring supports. Comparison studies are carried out to verify the correctness of the proposed method with published values (Leissa, 1973). The effect of different included angles, locations of ring supports and boundary conditions on the vibration behaviour of open circular cylindrical shells is investigated in the paper.

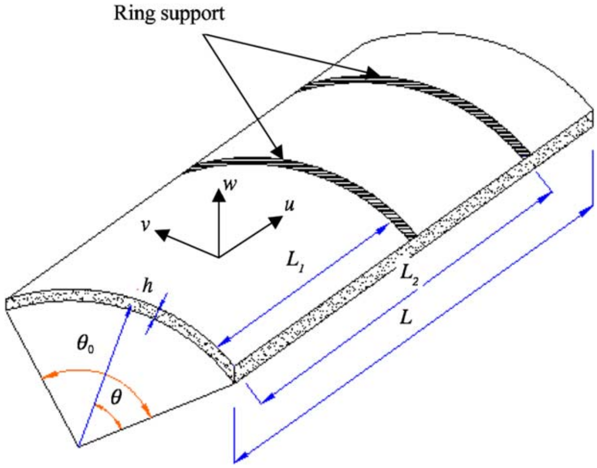

Fig. 1. Geometry and coordinate system for an open circular cylindrical shell with two intermediate ring supports.

2. Formulations

Consider an isotropic, open circular cylindrical shell with length L, included angle θ_0, uniform thickness h, midsurface radius R, Young's modulus E, Poisson's ratio υ and mass density ρ as shown in Fig. 1. The displacement fields on the midsurface of the open shell with reference to the coordinate system are denoted as $u(x,\theta,t)$, $v(x,\theta,t)$ and $w(x,\theta,t)$ in the x, θ and radial directions, respectively. The shell is assumed to be supported by $(q-1)$ intermediate ring supports that provide no elastic resistance other than radial restraint to the shell at the support locations, i.e., $w(x,\theta,t)=0$. Fig. 1 shows a typical shell with two intermediate ring supports located at distances L_1 and L_2 away from the left end of the shell. The two straight edges of the shell are assumed to be simply supported. The problem at hand is to determine the natural frequencies of the shell.

An analytical method based on the state-space technique was developed by Xiang et al. (2002) for the vibration analysis of circular cylindrical shells. This method is extended in this paper to study the vibration of open circular cylindrical shells with intermediate ring supports. For convenience and clarity, the method is described in details as follows.

2.1. Governing differential equations

The governing differential equations for the free vibration of a thin open circular cylindrical shell based on the Flügge shell theory can be written as (Leissa, 1973)

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{(1+\upsilon)}{2R}\frac{\partial^2 v}{\partial x\partial\theta}+\frac{\upsilon}{R}\frac{\partial w}{\partial x}+k\left[\frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u}{\partial\theta^2}-R\frac{\partial^3 w}{\partial x^3}+\frac{(1-\upsilon)}{2R}\frac{\partial^3 w}{\partial x\partial^2\theta}\right]$$
$$=\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{(1+\upsilon)}{2R}\frac{\partial^2 u}{\partial x\partial\theta}+\frac{(1-\upsilon)}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{1}{R^2}\frac{\partial w}{\partial\theta}+k\left[\frac{3(1-\upsilon)}{2}\frac{\partial^2 v}{\partial x^2}-\frac{(3-\upsilon)}{2}\frac{\partial^3 w}{\partial x^2\partial\theta}\right]$$
$$=\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 v}{\partial t^2} \tag{2}$$

$$\frac{\upsilon}{R}\frac{\partial u}{\partial x}+\frac{1}{R^2}\frac{\partial v}{\partial\theta}+\frac{1}{R^2}w$$
$$+k\left[R^2\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial\theta^2}+\frac{1}{R^2}\frac{\partial^4 w}{\partial\theta^4}-R\frac{\partial^3 u}{\partial x^3}+\frac{(1-\upsilon)}{2R}\frac{\partial^3 u}{\partial x\partial\theta^2}-\frac{(3-\upsilon)}{2}\frac{\partial^3 v}{\partial x^2\partial\theta}+\frac{1}{R^2}w+\frac{2}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right]$$
$$=-\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 w}{\partial t^2} \tag{3}$$

in which $k=h^2/(12R^2)$.

2.2. Solutions for the ith segment

The shell can be divided along the axial direction (x direction) into q segments at the locations of the ring supports. As the two straight edges of the shell are assumed to be simply supported, the displacement fields for the ith segment may be expressed as

$$u_i(x,\theta,t)=U_i(x)\sin\left(\frac{m\pi}{\theta_0}\theta\right)\cos\omega t \tag{4}$$

$$v_i(x,\theta,t)=V_i(x)\cos\left(\frac{m\pi}{\theta_0}\theta\right)\cos\omega t \tag{5}$$

$$w_i(x,\theta,t)=W_i(x)\sin\left(\frac{m\pi}{\theta_0}\theta\right)\cos\omega t \tag{6}$$

where the subscript $i(=1,2,3,\ldots,q)$ denotes the ith segment of the shell, $m(=0,1,2,\ldots,\infty)$ is the number of half-waves of a vibration mode in the circumferential direction, ω is the angular frequency of vibration, and $U_i(x)$, $V_i(x)$ and $W_i(x)$ are unknown functions to be determined. Note that $m=0$ corresponds to the axi-symmetric vibration mode for the shell. We restrict our study in this paper to $m>0$.

Employing the state-space technique, a homogenous differential equation system for the ith segment can be derived from Eqs. (1)–(3) and (4)–(6) after appropriate algebraic operations (Xiang et al., 2002)

$$\boldsymbol{\psi}_i'-\mathbf{H}_i\boldsymbol{\psi}_i=0 \tag{7}$$

in which

$$\boldsymbol{\psi}_i=[\,U_i\quad U_i'\quad V_i\quad V_i'\quad W_i\quad W_i'\quad W_i''\quad W_i'''\,]^{\mathrm{T}} \tag{8}$$

the prime (′) denotes the derivative with respect to x, and $\mathbf{H}_i$ is an 8×8 matrix with the following non-zero elements:

$$(H_i)_{12}=(H_i)_{34}=(H_i)_{56}=(H_i)_{67}=(H_i)_{78}=1 \tag{9}$$

$$(H_i)_{21}=\frac{m^2\pi^2(1-\upsilon)(1+k)}{2R^2\theta_0^2}-\frac{\rho(1-\upsilon^2)\omega^2}{E} \tag{10}$$

$$(H_i)_{24}=\frac{(1+\upsilon)m\pi}{2R\theta_0} \tag{11}$$

$$(H_i)_{26}=-\frac{\upsilon}{R}+\frac{(1-\upsilon)}{2R\theta_0^2}km^2\pi^2 \tag{12}$$

$$(H_i)_{28}=kR \tag{13}$$

$$(H_i)_{42}=\frac{-(1+\upsilon)m\pi}{R(1-\upsilon)(1+3k)\theta_0} \tag{14}$$

$$(H_i)_{43}=\frac{2m^2\pi^2}{R^2(1-\upsilon)(1+3k)\theta_0^2}-\frac{2\rho(1-\upsilon^2)\omega^2}{E(1-\upsilon)(1+3k)} \tag{15}$$

$$(H_i)_{45} = \frac{-2m\pi}{R^2(1-\upsilon)(1+3k)\theta_0} \tag{16}$$

$$(H_i)_{47} = \frac{mk(3-\upsilon)\pi}{(1-\upsilon)(1+3k)\theta_0} \tag{17}$$

$$(H_i)_{82} = -\frac{\upsilon}{kR^3(1-k)} + \frac{m^2(1-\upsilon)\pi^2}{R^3(1-k)\theta_0^2} + \frac{km^2(1-\upsilon)\pi^2}{2R^3(1-k)\theta_0^2} - \frac{\rho(1-\upsilon^2)\omega^2}{ER(1-k)} + \frac{m^2(1+\upsilon)\pi^2}{R^3(1-k)(1+3k)\theta_0^2} \tag{18}$$

$$(H_i)_{83} = \frac{m\pi}{kR^4(1-k)\theta_0} - \frac{2m^3\pi^3}{R^4(1-k)(1+3k)\theta_0^3} - \frac{2m\pi\rho(1-\upsilon^2)\omega^2}{ER^2(1-k)(1+3k)\theta_0} \tag{19}$$

$$(H_i)_{85} = -\frac{1}{kR^4(1-k)} - \frac{m^4\pi^4}{R^4(1-k)\theta_0^4} + \frac{2m^2\pi^2}{R^4(1-k)\theta_0^2} - \frac{1}{R^4(1-k)} + \frac{\rho\omega^2(1-\upsilon^2)}{kR^2E(1-k)} + \frac{2m^2\pi^2}{R^4(1-k)(1+3k)\theta_0^2} \tag{20}$$

$$(H_i)_{87} = \frac{2m^2\pi^2}{R^2(1-k)\theta_0^2} - \frac{\upsilon}{R^2(1-k)} + \frac{km^2(1-\upsilon)\pi^2}{2R^2(1-k)\theta_0^2} - \frac{km^2(3-\upsilon)\pi^2}{R^2(1-k)(1+3k)\theta_0^2} \tag{21}$$

The general solution of the homogenous differential equation system defined by Eq. (7) can be expressed as follows (Braun, 1993):

$$\boldsymbol{\psi}_i = \mathrm{e}^{\mathbf{H}_i x}\mathbf{c}_i \tag{22}$$

where $\mathbf{c}_i$ is an 8×1 constant column matrix that is to be determined using the boundary conditions and/or interface conditions between the shell segments, and $\mathrm{e}^{\mathbf{H}_i x}$ is a general matrix solution of Eq. (7) which can be obtained by

$$\mathrm{e}^{\mathbf{H}_i x} = \mathbf{Z}(x)\mathbf{Z}^{-1}(0) \tag{23}$$

in which $\mathbf{Z}(x)$ is a fundamental matrix solution of Eq. (7) and $\mathbf{Z}^{-1}(0)$ is the inverse of matrix $\mathbf{Z}(x)$ when the variable $x = 0$ (Braun, 1993; Xiang and Liew, 1996; Liew et al., 1996). Based on the solutions of the eigenvalues of matrix $\mathbf{H}_i$, the fundamental matrix solution of Eq. (7) can be formed as follows:

(1) if all eigenvalues of $\mathbf{H}_i$ are real and distinctive, i.e., $r_1, r_2, \ldots, r_8$ and the corresponding eigenvectors are $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_8$, the fundamental solution of Eq. (7) is given by

$$\mathbf{Z}(x) = [\,\mathbf{Z}_1(x) \quad \mathbf{Z}_2(x) \quad \mathbf{Z}_3(x) \quad \mathbf{Z}_4(x) \quad \mathbf{Z}_5(x) \quad \mathbf{Z}_6(x) \quad \mathbf{Z}_7(x) \quad \mathbf{Z}_8(x)\,] \tag{24}$$

where $\mathbf{Z}_j(x)$= where $\mathbf{Z}_j(x) = \mathrm{e}^{r_j x}\mathbf{s}_j$, $j = 1, 2, \ldots, 8$.

(2) if the matrix $\mathbf{H}_i$ is of n (even number) real and distinctive eigenvalues and $(8-n)/2$ pairs of conjugate complex eigenvalues, the contribution of the real and distinctive eigenvalues and their corresponding eigenvectors to the fundamental solution of Eq. (7) is the same as the term $\mathbf{Z}_j(x) = \mathrm{e}^{r_j x}\mathbf{s}_j$ in Eq. (24). While for a pair of conjugate eigenvalues $r_j = r_j^a + ir_j^b$ and $r_{j+1} = r_j^a - ir_j^b$ with corresponding eigenvectors $\mathbf{s}_j = \mathbf{s}_j^a + i\mathbf{s}_j^b$ and $\mathbf{s}_{j+1} = \mathbf{s}_j^a - i\mathbf{s}_j^b$, their contribution to the fundamental solution of Eq. (7) can be expressed as

$$\mathbf{Z}_j(x) = \mathrm{e}^{r_j^a x}(\mathbf{s}_j^a \cos r_j^b x - \mathbf{s}_j^b \sin r_j^b x) \tag{25}$$

$$\mathbf{Z}_{j+1}(x) = \mathrm{e}^{r_j^a x}(\mathbf{s}_j^a \sin r_j^b x + \mathbf{s}_j^b \cos r_j^b x) \tag{26}$$

For other cases of eigenvalues of $\mathbf{H}_i$ (such as repetitive eigenvalues), the formation of $\mathbf{Z}(x)$ was discussed in Braun (1993). In this study, we have only encountered the two cases of eigenvalues of $\mathbf{H}_i$ as discussed above.

2.3. Boundary and interface conditions

It is well known that there are four types of simply supported and four types of clamped boundary conditions at an edge of a circular cylindrical shell. Although we can obtain exact vibration solutions for an open circular cylindrical shell with various combinations of circumferential end support conditions, in this paper three typical boundary conditions are considered and are defined as follows:

(1) Simply supported or shear diaphragm (S):

$$w_i = (M_x)_i = (N_x)_i = v_i = 0 \tag{27}$$

(2) Free (F):

$$(N_x)_i = (N_{x\theta})_i + \frac{(M_{x\theta})_i}{R} = (Q_x)_i + \frac{1}{R}\frac{\partial (M_{x\theta})_i}{\partial \theta} = (M_x)_i = 0 \tag{28}$$

(3) Clamped (C):

$$u_i = v_i = w_i = \frac{\partial w_i}{\partial x} = 0 \tag{29}$$

where i takes the value 1 or q, and the force and moment resultants based on the Flügge shell theory are given as (Leissa, 1973)

$$N_x = \frac{Eh}{(1-\upsilon^2)}\left[\varepsilon_x + \upsilon\varepsilon_\theta + \frac{h^2}{12R}\kappa_x\right] \tag{30}$$

$$N_\theta = \frac{Eh}{(1-\upsilon^2)}\left[\varepsilon_\theta + \upsilon\varepsilon_x - \frac{h^2}{12R}\left(\kappa_\theta - \frac{1}{R}\varepsilon_\theta\right)\right] \tag{31}$$

$$N_{x\theta} = \frac{Eh}{2(1+\upsilon)}\left(\varepsilon_{x\theta} + \frac{h^2}{24R}\tau\right) \tag{32}$$

$$M_x = \frac{Eh^3}{12(1-\upsilon^2)}\left(\kappa_x + \upsilon\kappa_\theta + \frac{1}{R}\varepsilon_x\right) \tag{33}$$

$$M_{x\theta} = \frac{Eh^3}{24(1+\upsilon)}\tau \tag{34}$$

$$M_{\theta x} = \frac{Eh^3}{24(1+\upsilon)}\left(\tau - \frac{1}{R}\varepsilon_{x\theta}\right) \tag{35}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{R}\frac{\partial M_{\theta x}}{\partial \theta} \tag{36}$$

and the strain, curvature and twist of middle surface terms are relating to displacement fields by

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{37}$$

$$\varepsilon_\theta = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right) \tag{38}$$

$$\varepsilon_{x\theta} = \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \tag{39}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{40}$$

$$\kappa_\theta = \frac{1}{R^2}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right) \tag{41}$$

$$\tau = -\frac{2}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) \tag{42}$$

To ensure the equilibrium and compatibility along the interface between the ith and the $(i+1)$th segments, the displacement fields and force resultants at the interface must satisfy the follow conditions:

$$w_i = 0 \tag{43}$$

$$w_{i+1} = 0 \tag{44}$$

$$u_i = u_{i+1} \tag{45}$$

$$v_i = v_{i+1} \tag{46}$$

$$\frac{\partial w_i}{\partial x} = \frac{\partial w_{i+1}}{\partial x} \tag{47}$$

$$(M_x)_i = (M_x)_{i+1} \tag{48}$$

$$(N_x)_i = (N_x)_{i+1} \tag{49}$$

$$\left(N_{x\theta} + \frac{M_{x\theta}}{R}\right)_i = \left(N_{x\theta} + \frac{M_{x\theta}}{R}\right)_{i+1} \tag{50}$$

2.4. Assembling segments to shell

The constant column matrix $\mathbf{c}_i$ in Eq. (22) can be determined by applying the shell boundary conditions and interface conditions between shell segments. For example, for an open shell with simply supported left circumferential edge, clamped right circumferential edge and one intermediate ring support, there are four boundary equations contributed from the left circumferential edge (Eq. (27)), four from the right circumferential edge (Eq. (29)) and eight equations from the interface between the two shell segments (Eqs. (43)–(50)). All the 16 boundary/interface equations can be expressed in terms of the constant column matrices $\mathbf{c}_1$ and $\mathbf{c}_2$ through their relationship with the displacement terms $\boldsymbol{\psi}_i = [\,U_i \quad U_i' \quad V_i \quad V_i' \quad W_i \quad W_i' \quad W_i'' \quad W_i'''\,]^{\mathrm{T}}$.

In view of Eq. (22), a homogeneous system of equations can be derived by implementing the boundary conditions of the shell (see Eqs. (27)–(29)) and the interface conditions between two segments (Eqs. (43)–(50)) when assembling the segments to form the whole shell. We have

$$\mathbf{Kc} = \mathbf{0} \tag{51}$$

where $\mathbf{K}$ is an $8q \times 8q$ matrix and $\mathbf{c} = [\,\mathbf{c}_1^{\mathrm{T}} \quad \mathbf{c}_2^{\mathrm{T}} \quad \cdots \quad \mathbf{c}_q^{\mathrm{T}}\,]^{\mathrm{T}}$ is an $8q \times 1$ column matrix. The angular frequency ω is evaluated by setting the determinant of $\mathbf{K}$ in Eq. (51) to zero.

An iteration process proposed in Xiang and Liew (1996) and Liew et al. (1996) is employed to evaluate the eigenvalue (ω). And the eigenvalue equation Eq. (51) is highly sensible to the numerical accuracy of the

Table 1
Comparison study for frequency parameters λ of simply supported open circular cylindrical shells ($R/h = 500, \upsilon = 0.3$)

$m\pi/\theta_0$	Sources	$L/(nR)$					
		0.1	0.25	1	4	20	100
1/3	Leissa (1973)	1.11097	0.957324	0.935744	0.422168	0.0514304	0.00264474
	Present	1.11097	0.957324	0.935744	0.422169	0.0514304	0.00264474
1/2	Leissa (1973)	1.11089	0.956487	0.919705	0.381374	0.0368061	0.00168460
	Present	1.11090	0.956487	0.919705	0.381375	0.0368061	0.00168460
2/3	Leissa (1973)	1.110790	0.955317	0.898558	0.337723	0.0271812	0.00118447
	Present	1.110791	0.955318	0.898558	0.337723	0.0271812	0.00118447

Table 2
Frequency parameters λ for SS and CC open circular cylindrical shells with one intermediate ring support ($L_1/L = 1/2, h/R = 0.01$)

L/R	θ_0	n	SS			CC		
			m			m		
			1	2	3	1	2	3
1	30°	1	0.512894	0.562968	1.049530	0.565426	0.604321	1.072288
		2	0.538311	0.601647	1.072146	0.574484	0.650365	1.100237
		3	0.954470	0.999467	1.420612	1.023446	1.080656	1.489347
10	30°	1	0.101257	0.412510	0.932115	0.102086	0.412570	0.932137
		2	0.102086	0.412570	0.932137	0.103342	0.412634	0.935615
		3	0.111253	0.416003	0.935523	0.114898	0.416242	0.935709
1	90°	1	0.773258	0.592130	0.512894	0.786943	0.632110	0.565426
		2	0.870170	0.690161	0.538311	0.883745	0.710941	0.574484
		3	1.040087	0.999101	0.954470	1.098345	1.061737	1.023446
10	90°	1	0.076378	0.048443	0.101257	0.101876	0.054440	0.102086
		2	0.091920	0.054374	0.102086	0.113555	0.062613	0.103342
		3	0.211182	0.094879	0.111253	0.224751	0.107871	0.114898
1	180°	1	0.591610	0.773258	0.665326	0.922774	0.786943	0.692953
		2	0.901726	0.870170	0.782475	0.945205	0.883745	0.798956
		3	0.935186	1.040087	1.021531	1.109402	1.098345	1.081671
10	180°	1	0.121177	0.076378	0.044274	0.151012	0.101876	0.059272
		2	0.186356	0.091920	0.058405	0.211326	0.113555	0.074525
		3	0.346533	0.211182	0.136770	0.354709	0.224751	0.153982
1	270°	1	0.394407	0.788814	0.773258	0.957438	0.874018	0.786943
		2	0.936416	0.860126	0.870170	0.969480	0.928505	0.883745
		3	0.948513	0.917248	1.040087	1.111541	1.106457	1.098345
10	270°	1	0.124951	0.116901	0.076378	0.161841	0.141316	0.101876
		2	0.254677	0.135624	0.091920	0.266837	0.164580	0.113555
		3	0.394407	0.284872	0.211182	0.428896	0.297167	0.224751
1	360°	1	0.295805	0.591610	0.838212	0.961777	0.922774	0.850530
		2	0.949628	0.901726	0.887416	0.989402	0.945205	0.918622
		3	0.953284	0.935186	0.906746	1.112297	1.109402	1.104695
10	360°	1	0.133204	0.121177	0.112281	0.169094	0.151012	0.135454
		2	0.293340	0.186356	0.116337	0.293615	0.211326	0.145323
		3	0.295805	0.346533	0.261427	0.432270	0.354709	0.274593

Table 3
Frequency parameters λ for CF and FF open circular cylindrical shells with one intermediate ring support ($L_1/L = 1/2, h/R = 0.01$)

L/R	θ_0	n	CF			FF		
			m			m		
			1	2	3	1	2	3
1	30°	1	0.204323	0.437836	0.950666	0.119401	0.431302	0.948317
		2	0.566543	0.623428	1.084381	0.256865	0.445117	0.953225
		3	0.693065	0.730025	1.193543	0.677106	0.702879	1.172621
10	30°	1	0.099743	0.410723	0.9292712	0.099695	0.410722	0.9292711
		2	0.102412	0.412600	0.932149	0.099796	0.411063	0.9292713
		3	0.104154	0.413424	0.932769	0.103116	0.413365	0.9327479
1	90°	1	0.447162	0.270482	0.204323	0.013937	0.058593	0.119401
		2	0.846298	0.657460	0.566543	0.332188	0.306874	0.256865
		3	0.925565	0.807133	0.693065	0.920712	0.781013	0.677106
10	90°	1	0.020794	0.042535	0.099743	0.007869	0.042139	0.099695
		2	0.106686	0.055225	0.102412	0.029064	0.042957	0.099796
		3	0.118529	0.063507	0.104154	0.115557	0.056110	0.103116
1	180°	1	0.616532	0.447162	0.342852	0.272137	0.013937	0.034728
		2	0.936973	0.846298	0.738011	0.877678	0.332188	0.329426
		3	0.961990	0.925565	0.870862	0.960294	0.920712	0.854434
10	180°	1	0.044439	0.020794	0.024067	0.061015	0.007869	0.022056
		2	0.151430	0.106686	0.061148	0.159533	0.029064	0.026062
		3	0.237409	0.118529	0.076307	0.261700	0.115557	0.063283
1	270°	1	0.701962	0.548414	0.447162	0.001248	0.003132	0.013937
		2	0.951941	0.913607	0.846298	0.212997	0.305764	0.332188
		3	0.968714	0.952359	0.925565	0.902124	0.884177	0.920712
10	270°	1	0.054831	0.033912	0.020794	0.000957	0.001853	0.007869
		2	0.164090	0.141717	0.106686	0.060028	0.049086	0.029064
		3	0.302723	0.182540	0.118529	0.191179	0.146006	0.115557
1	360°	1	0.752607	0.616532	0.519430	0.001274	0.272137	0.005369
		2	0.956663	0.936973	0.898873	0.170822	0.877678	0.316205
		3	0.970940	0.961990	0.946547	0.919934	0.960294	0.900175
10	360°	1	0.060562	0.044439	0.029456	0.001063	0.061015	0.003082
		2	0.171339	0.151430	0.135750	0.051080	0.159533	0.012774
		3	0.324153	0.237409	0.160611	0.217853	0.261700	0.140342

software package employed, especially when the shell thickness to radius ratio (h/R) is small. The mathematical software package PARI/GP (Batut et al., 2003) was employed in the computation of the eigenvalues as the software can handle numerical calculation with arbitrary number of significant digits with only the limitation of the computer CPU speed and memory capacity. We have used up to 1024 significant digits in some of the computations presented in the paper.

3. Results and discussions

The proposed analytical method is applied in this section to obtain exact vibration frequencies for open circular cylindrical shells with intermediate ring supports, different combinations of boundary conditions

Table 4
Frequency parameters λ for SS and CC open circular cylindrical shells with two intermediate ring supports ($L_1/L = 1/3, L_2/L = 2/3$, $h/R = 0.01$)

L/R	θ_0	n	SS			CC		
			m			m		
			1	2	3	1	2	3
1	30°	1	0.759580	0.759197	1.204131	0.767825	0.801577	1.237019
		2	0.761694	0.797759	1.236689	0.844263	0.900830	1.314271
		3	0.764157	0.889846	1.313058	0.856251	0.955737	1.361389
10	30°	1	0.104578	0.413948	0.933533	0.106097	0.414048	0.933572
		2	0.106093	0.414048	0.933572	0.109931	0.414265	0.933651
		3	0.109908	0.414265	0.933651	0.112567	0.414383	0.933693
1	90°	1	0.902457	0.788038	0.759580	0.919939	0.820420	0.767825
		2	0.949608	0.847981	0.761694	1.000132	0.915748	0.844263
		3	0.955040	0.859686	0.764157	1.017114	0.918221	0.856251
10	90°	1	0.148020	0.065755	0.104578	0.149705	0.073694	0.106097
		2	0.148317	0.073454	0.106093	0.176997	0.090175	0.109931
		3	0.153527	0.089101	0.109908	0.182174	0.098505	0.112567
1	180°	1	0.591610	0.902457	0.828731	0.988715	0.919939	0.859059
		2	0.977297	0.949608	0.902781	1.051916	1.000132	0.957710
		3	0.993618	0.955040	0.908413	1.065875	1.017114	0.968432
10	180°	1	0.148450	0.148020	0.084408	0.170787	0.149705	0.096814
		2	0.226886	0.148317	0.095086	0.251018	0.176997	0.120152
		3	0.327098	0.153527	0.113925	0.338639	0.182174	0.126982
1	270°	1	0.394407	0.788814	0.902457	1.000151	0.969997	0.919939
		2	0.982693	0.969856	0.949608	1.058742	1.036773	1.000132
		3	1.001485	0.969887	0.955040	1.100930	1.042535	1.017114
10	270°	1	0.140036	0.155054	0.148020	0.173896	0.167978	0.149705
		2	0.274725	0.194760	0.148317	0.286319	0.222652	0.176997
		3	0.394407	0.252584	0.153527	0.420958	0.271260	0.182174
1	360°	1	0.295805	0.591610	0.887416	1.003701	0.988715	0.958159
		2	0.984602	0.977297	0.952236	1.061153	1.051916	1.025550
		3	1.004297	0.993618	0.965469	1.117671	1.065875	1.036953
10	360°	1	0.141922	0.148450	0.155720	0.176943	0.170787	0.165407
		2	0.295805	0.226886	0.181987	0.303814	0.251018	0.210485
		3	0.304903	0.327098	0.222077	0.435204	0.338639	0.243496

and various included angles. For convenience, a two-letter symbol is used to describe the end boundary conditions for the two curved edges. For example, the symbol CF denotes a shell having clamped and free edge conditions at $x = 0$ and $x = L$, respectively. The vibration frequency ω is expressed in terms of a non-dimensional frequency parameter $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$. The Poisson ratio υ takes the value 0.3 in this study.

3.1. Verification of solution method

A comparison study is carried out for frequency parameters λ of a simply supported open cylindrical shell without intermediate support obtained by Leissa (1973) and the present analytical approach (see Table 1). The value of m in Table 1 denotes the number of half-waves of a vibration mode in the circumferential

Table 5
Frequency parameters λ for CF and FF open circular cylindrical shells with two intermediate ring supports ($L_1/L = 1/3, L_2/L = 2/3, h/R = 0.01$)

L/R	θ_0	n	CF			FF		
			m			m		
			1	2	3	1	2	3
1	30°	1	0.312092	0.479965	0.984006	0.268280	0.478541	0.983230
		2	0.764875	0.815222	1.248490	0.348076	0.481417	0.984794
		3	0.849843	0.929864	1.340230	0.761749	0.835305	1.266511
10	30°	1	0.100100	0.410836	0.929310	0.100071	0.410836	0.929310
		2	0.106431	0.414085	0.933586	0.100150	0.410837	0.929310
		3	0.110867	0.414336	0.935483	0.106865	0.414145	0.933609
1	90°	1	0.563602	0.397150	0.312092	0.210790	0.267598	0.268080
		2	0.928175	0.816708	0.764875	0.502138	0.451354	0.348076
		3	1.008377	0.916463	0.849843	0.935166	0.818733	0.761749
10	90°	1	0.039176	0.044080	0.100110	0.037234	0.043662	0.100071
		2	0.149321	0.074859	0.106431	0.041034	0.044512	0.100150
		3	0.179045	0.092455	0.110867	0.149008	0.076189	0.106865
1	180°	1	0.693330	0.563602	0.470470	0.080113	0.210790	0.259556
		2	0.982939	0.928175	0.862487	0.407833	0.502138	0.494831
		3	1.042030	1.008377	0.962299	0.895231	0.935166	0.873281
10	180°	1	0.072986	0.039176	0.030029	0.056705	0.037234	0.028499
		2	0.165891	0.149321	0.099019	0.086921	0.041034	0.031546
		3	0.270057	0.179045	0.123300	0.162616	0.149008	0.101588
1	270°	1	0.757826	0.642453	0.563602	0.037735	0.129071	0.210790
		2	0.992104	0.968880	0.928175	0.314198	0.461279	0.502138
		3	1.043361	1.034366	1.008377	0.914830	0.903713	0.935166
10	270°	1	0.077588	0.061014	0.039176	0.046433	0.055468	0.037234
		2	0.170923	0.164457	0.149321	0.087216	0.066358	0.041034
		3	0.318347	0.233298	0.179045	0.188368	0.160881	0.149008
1	360°	1	0.796833	0.693330	0.620553	0.021607	0.080113	0.152929
		2	0.994695	0.982939	0.960065	0.249208	0.407833	0.477653
		3	1.040310	1.042030	1.028811	0.929367	0.895231	0.921983
10	360°	1	0.076547	0.072986	0.054677	0.035650	0.056705	0.052406
		2	0.174994	0.165891	0.162745	0.073427	0.086921	0.056848
		3	0.332495	0.270057	0.218413	0.215996	0.162616	0.160086

direction. It is observed that an excellent agreement is achieved between the present results and the ones by Leissa (1973). The comparison study confirms the correctness of the proposed analytical method for the vibration analysis of open circular cylindrical shells.

3.2. Vibration of open shells without and with ring supports

The proposed method is able to obtain exact solutions for vibration of open cylindrical shells with multiple intermediate ring supports. For the purpose of providing benchmark results, Tables 2–5 present the exact frequency parameters λ for open circular cylindrical shells with one and two ring supports at various included angles. Four different combinations of shell end boundary conditions are considered which are

SS, CC, CF and FF, respectively. The shell length to radius ratio L/R is set to be 1 and 10, the thickness to radius ratio is fixed at $h/R = 0.01$ and the included angle θ_0 varies from 30° to 360°, respectively. For a shell with one intermediate ring support, the location of the ring support is at the middle of the shell. The locations of the ring supports are at one-third and two-third length of the shell if two intermediate ring supports are considered. The value of m represents the number of circumferential half-waves and n is the mode sequence number for a given m value. As expected, shells with clamped–clamped end supports (CC shells) have higher frequency parameters when compared with shells of other edge support conditions. An increase in the length to radius ratio L/R from 1 to 10 will lead to a decrease in the frequency parameters. It is observed that the influence of end boundary conditions on the frequency parameters decreases as the length to radius ratio L/R increases. The variation of the frequency parameters is more sensitive to smaller included angle θ_0 when $L/R = 10$ and the circumferential wave number m is fixed. It is also found that the frequency

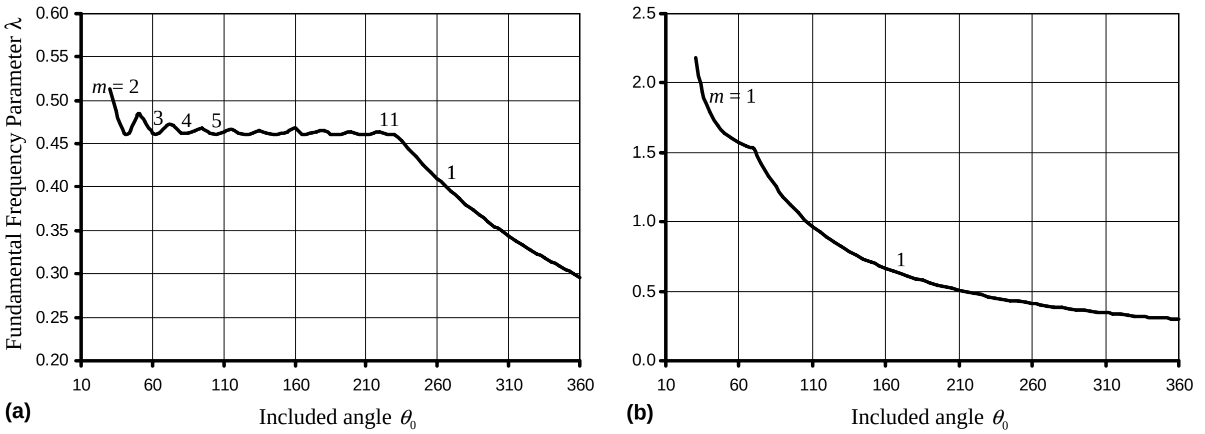

Fig. 2. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for an SS open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 1$. (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

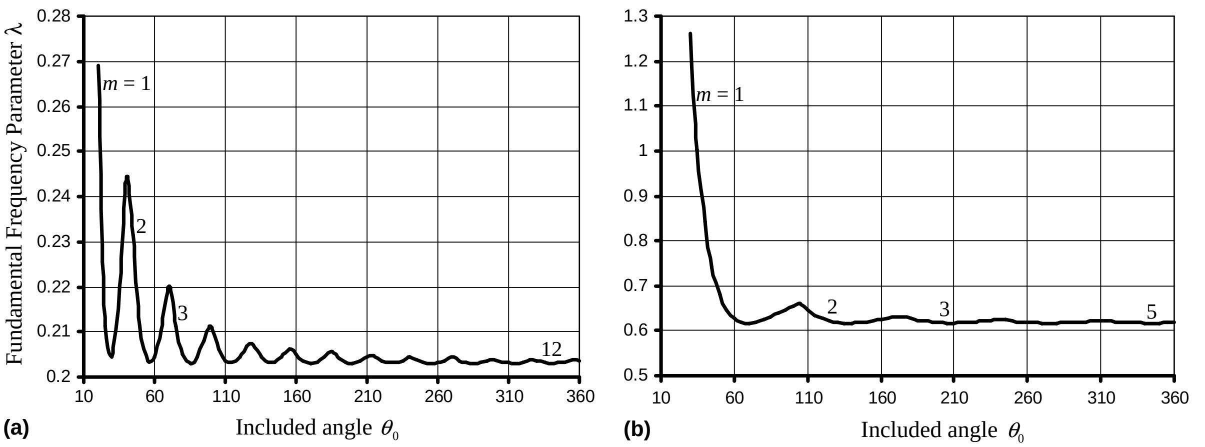

Fig. 3. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for a CF open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 1$. (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

parameters may increase or decrease as the number of circumferential half-waves m increases. Therefore, the lowest frequencies in Tables 2–5 may not be the fundamental frequencies for the considered cases.

The effect of the presence of the intermediate ring supports on the frequency parameter λ is examined. It is observed that the presence of the intermediate ring supports will increase the stiffness of the open shell that will lead to higher frequency parameters. Shells with two intermediate ring supports have higher frequency parameters when comparing shells with no ring support or one ring support.

3.3. Effect of included angle on fundamental frequency parameter of open shells

The fundamental frequency parameter of an open shell is calculated by increasing the values of the number of circumferential half-waves m until the minimum frequency parameter is obtained. Figs. 2–7 present

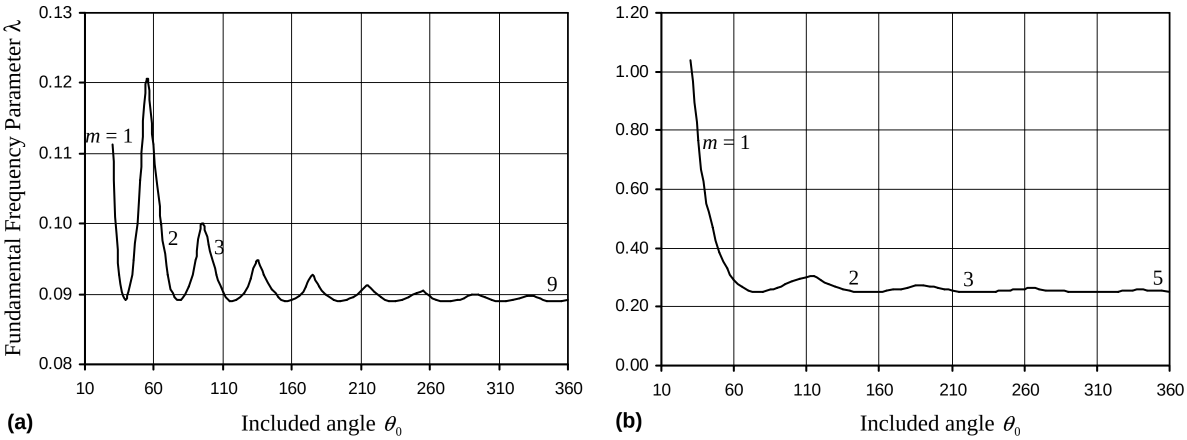

Fig. 4. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for a SS open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 5$. (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

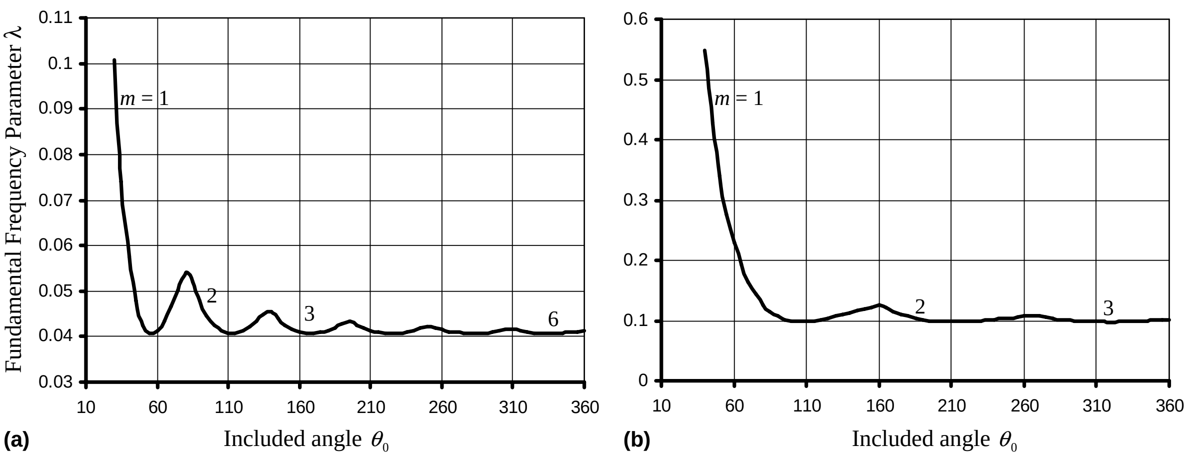

Fig. 5. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for a CF open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 5$. (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

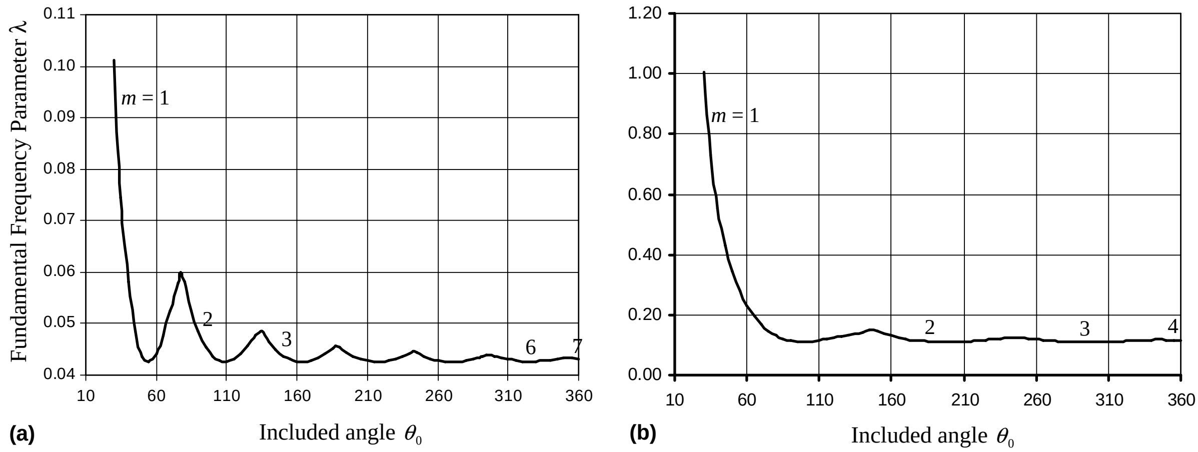

Fig. 6. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for a SS open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 10$ (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

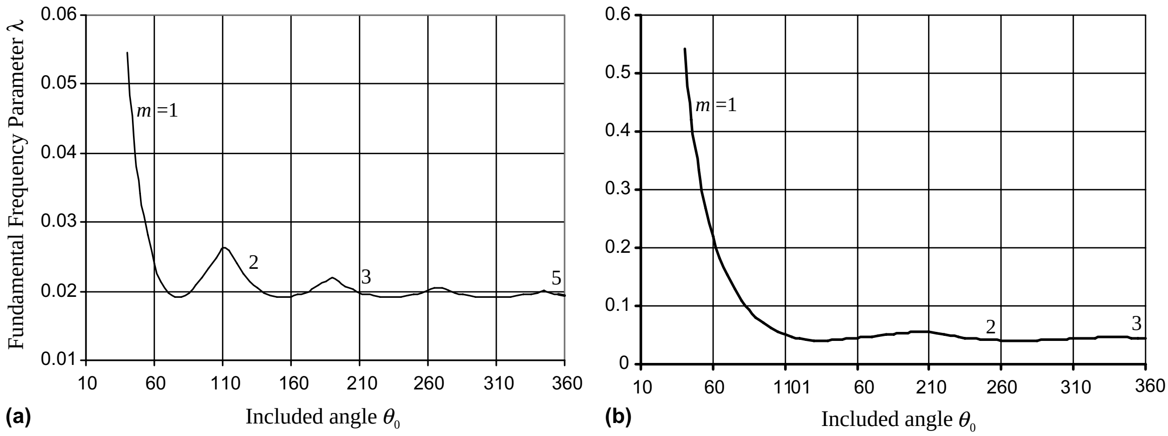

Fig. 7. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus included angle θ_0 for a CF open shell having one intermediate ring support at the middle of the shell and with the length to radius ratio $L/R = 10$. (a) $h/R = 0.01$ and (b) $h/R = 0.1$.

the fundamental frequency parameter λ versus included angle θ_0 for SS and CF open circular cylindrical shells with thickness to radius ratio $h/R =$ 0.1 and 0.01 and shell length to radius ratio $L/R = 1$, 5 and 10, respectively. One intermediate ring support is considered and the location of ring support is at the middle of the shell. The included angle θ_0 varies from 30° to 360° with an increment of 5°. It is intended to examine the effect of a varying included angle θ_0 on the fundamental frequency parameter while keeping the thickness and length of the shell constant.

It is observed that the increase of the thickness ratio h/R from 0.01 to 0.1 will lead to an increase in fundamental frequency parameters. It is because the increase of the thickness ratio will result in an increase in the bending stiffness of the open shell. There are more kink points on the graphs for shells with $h/R = 0.01$ than for shells with $h/R = 0.1$. These kink points represent such values of the included angle θ_0 where the mode shape switch occurs. The number of circumferential half-waves m is marked in the graphs.

The fundamental frequency parameter for an SS shell with $L/R = 1$ and $h/R = 0.01$ decreases initially and then maintains a small variation when the included angle θ_0 changes from 30° to 230° (see Fig. 2(a)). The frequency parameter decreases monotonically as the included angle θ_0 increases from 230° to 360°. However, for the same shell but with the thickness ratio $h/R = 0.1$, the variation of the fundamental frequency parameter against the included angle θ_0 is quite different (see Fig. 2(b)). The frequency parameter decreases sharply as the included angle θ_0 changes from 30° to 160° and then the rate of decrease is reduced when the included angle θ_0 increases further. For SS and CF shells with larger length to radius ratios ($L/R = 5$ and 10), the fundamental frequency parameter varies significantly for $\theta_0 < 100°$ and then oscillates about a constant value when θ_0 increases from 100° to 360°, as shown in Figs. 4–7.

Fig. 3 shows that the variation of the fundamental frequency parameter versus the included angle θ_0 for a CF shell with $L/R = 1$. The frequency parameter for this case shows a tendency that is quite different from its SS counterpart. The frequency parameter for the CF shell with $h/R = 0.01$ oscillates significantly as the included angle θ_0 increases from 30° to 110° and then varies within a small range as the included angle increases further.

3.4. Effect of location of ring support on fundamental frequency parameter of open shells

The effect of the presence of an intermediate ring support at different locations on the fundamental frequencies of SS and CF shells is illustrated in Fig. 8. The thickness to radius ratio $h/R = 0.01$, length to radius ratio $L/R = 5$ and included angle $\theta_0 = 90°$ are used in the calculation, respectively. One intermediate ring support is considered and the location parameter of the ring support L_1/L varies along the axial direction of the open shell from 0.01 to 0.5 with increment of 0.01 for the SS shell and from 0.02 to 0.98 with increment of 0.02 for the CF shell, respectively.

The location of the ring support has a significant effect on the fundamental frequency parameter and the influence varies with the end boundary conditions. The frequency parameter increases monotonically as the location of the ring support moves from the end of the shell to the middle. For an open cylindrical shell with symmetrical end boundary conditions, such as an SS shell, it is found that the optimal locations of an intermediate ring support, which will produce the maximum fundamental frequency, is at the middle of the open shell. The fundamental frequency parameter for a CF shell shows similar trends as for the SS shell.

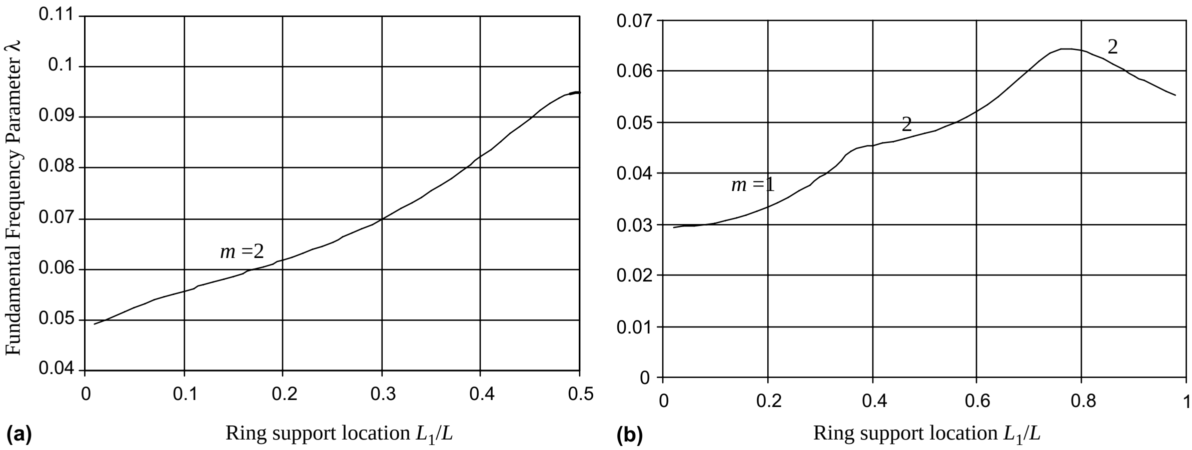

Fig. 8. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus ring support location L_1/L for SS and CF open cylindrical shells having one intermediate ring support with length to radius ratio $L/R = 5$, thickness ratio $h/R = 0.01$ and $\theta_0 = 90°$. (a) SS and (b) CF.

However, the fundamental frequency of the CF shell versus the location of the ring support is not symmetrical about the location of $L_1/L = 0.5$ due to the unsymmetrical end boundary conditions. The number of circumferential half-waves m is marked in the graphs.

3.5. Effect of shell length to radius ratio on fundamental frequency parameter of open shells

Figs. 9 and 10 present the fundamental frequency parameters against the shell length to radius ratio for SS and CC open shells with one intermediate ring support at the location of $L_1/L = 1/3$ and the included angle $\theta_0 = 90°$. The shell thickness to radius ratio h/R is set to be 0.005 and 0.1. It is observed that the

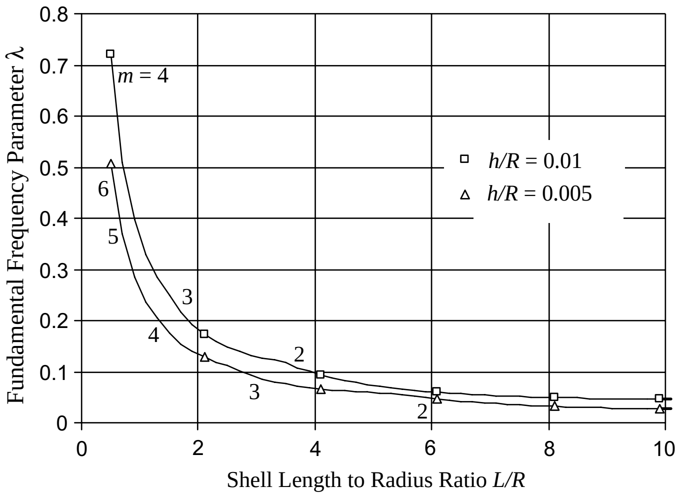

Fig. 9. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus shell length to radius ratio L/R for SS open cylindrical shells having one intermediate ring support at the location of $L_1L = 1/3$ and with included angle $\theta_0 = 90°$.

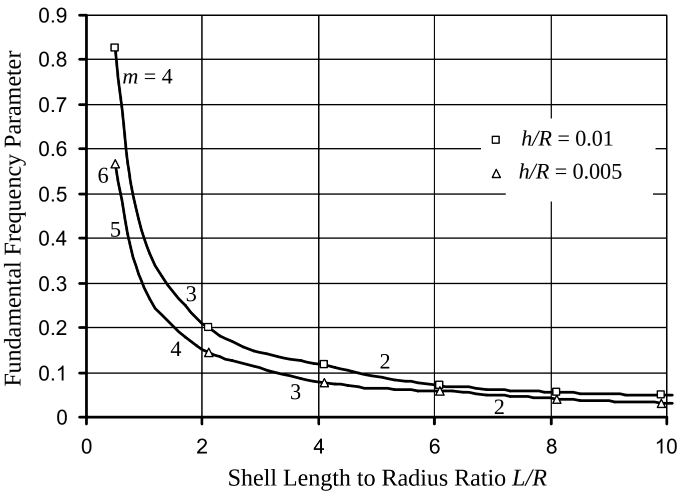

Fig. 10. Fundamental frequency parameter $\lambda = \omega R\sqrt{\rho(1-\upsilon^2)/E}$ versus shell length to radius ratio L/R for CC open cylindrical shells having one intermediate ring support at the location of $L_1L = 1/3$ and with included angle $\theta_0 = 90°$.

fundamental frequency parameters for both SS and CC shells decrease monotonically as the shell length to radius ratio L/R increases from 0.5 to 10. The number of circumferential half-waves is marked on the graphs and there are more circumferential half-waves in the vibration modes when the shell length to radius ratio L/R and the shell thickness to radius ratio h/R are smaller.

4. Conclusions

This paper presents an analytical solution procedure and exact vibration solutions for open circular cylindrical shells with various combinations of end boundary conditions and intermediate ring supports. The Flügge thin shell theory is employed and the state-space technique is applied to derive the solutions for the shell vibration problems. The proposed method has been verified through the comparison of present results against published solutions. The influence of the included angle, the shell length to radius ratio and the thickness to radius ratio on the frequency parameters of open circular cylindrical shells is examined. Exact frequency parameters are also presented in tabular form for easy reference when required as benchmark values for researchers to valid their numerical approaches for such shell vibration problems.

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